JOURNAL OF APPROXIMATION THEORY 62, 133–144 (1990)

Almost Cardinal Spline Interpolation

NUR ARAD* AND NIRA DYN

School of Mathematical Sciences, Sackler Faculty of Exact Sciences, Tel-Aviv University, Tel-Aviv, Israel

Communicated by Allan Pinkus

Received August 11, 1988; revised October 28, 1988

Interpolation of a doubly infinite sequence of data by spline functions is studied. When the interpolation points and the knots of the interpolating splines are characterized by a periodic behavior, the interpolating problem is called Cardinal Interpolation. This work extends known results on Cardinal Interpolation to the "almost cardinal" case, where the interpolation is cardinal except for a finite number of interpolation points and knots. In passing from the cardinal to the "almost cardinal" case, the "invariance under translation" property of the interpolating spaces is lost. Thus classical arguments used in solving the cardinal case do not apply. Instead we use the intimate connection between the interpolating "almost cardinal splines" and Oscillatory Matrices. The main conclusion of this work is that a wide range of Almost Cardinal Interpolation Problems have the same type of solution as the corresponding Cardinal Interpolation Problem. © 1990 Academic Press, Inc.

1. INTRODUCTION

This paper deals with a certain generalization of known results on Cardinal Spline Interpolation. The characteristic feature of Cardinal Spline Interpolation Problems is the periodicity of the knot sequence which defines the spline functions and the placement of the interpolation points. The interpolation problem which we use as a starting point is the following:

Let $\bar{\triangle} = \{\xi_0, \xi_1, ..., \xi_{\hat{c}}\}$, such that $0 = \xi_0 < \xi_1 < \cdots < \xi_{\hat{c}} = 1$ and $\hat{c}$ a positive integer, be given. We extend periodically the partition $\bar{\triangle}$ of the unit interval to the infinite line $\mathbb{R}$, and denote the resulting partition by $\triangle = \{t_k\}_{k \in \mathbb{Z}}$. The partition $\triangle$ and a natural number r define spaces of spline functions as follows: Given $n \geqslant 2r - 1$,

$$S_{n,r,\triangle} = \{f(x) \mid f_{[t_k, t_{k+1}]} \in \prod_n, f(x) \in C^{n-r}(\mathbb{R})\},$$

* Also at the Open University, Tel Aviv, Israel. The following is part of this author's MSc thesis.

0021-9045/90 $3.00

where $\prod_n$ is the space of algebraic polynomials of degree at most n. The Cardinal Interpolation Problem on $S_{n,r,\Delta}$ is the following class of problems:

Given r bounded sequences $\{f_k^{(j)}\}_{k \in \mathbb{Z}}$, $j = 0, 1, ..., r-1$, and a sequence $\alpha = \{\alpha_k\}_{k \in \mathbb{Z}}$ satisfying either

(a) $\alpha_k = 0$ for all k or

(b) $0 < \alpha_k < t_k - t_{k-1}$ and $\alpha_{k+\ell} = \alpha_k$ for all k,

find a unique bounded function $S_{n,r,\Delta}$ that satisfies $f^{(j)}(t_k + \alpha_k) = f_k^{(j)}$ for all $k \in \mathbb{Z}$ and $j = 0, 1, ..., r-1$. Some special cases are:

(I) $\ell = 1$, $r = 1$, $\alpha_k = 0$. This is the classical Lagrange Cardinal Interpolation Problem first considered by Schoenberg in [5]. In this case

$$S_{n,1,\mathbb{Z}} = S_n = \{f(x) \mid f_{[k,k+1]} \in \prod_n, f(x) \in C^{n-1}(\mathbb{R})\}$$

and the interpolation problem amounts to finding a unique bounded function $f(x) \in S_n$ satisfying $f(k) = f_k$ for all $k \in \mathbb{Z}$, for any bounded sequence $\{f_k\}_{k \in \mathbb{Z}}$.

(II) $\ell = 1$, $\alpha_k = 0$. In this case we relax the continuity conditions on functions in S_n, and in turn interpolate bounded Hermite data at the integers by bounded functions in $S_{n,r,\mathbb{Z}}$.

(III) $r = 1$, $\alpha_k = 0$. This case was first considered by Micchelli in [4].

(IV) $\ell = 1$, $\alpha_k = \frac{1}{2}$. This is Cardinal Lagrange Interpolation at the mid-points between the integers.

So much for Cardinal Spline Interpolation.

Generalizing the known results on the above Cardinal Interpolation Problem, we solve here a closely related "Almost Cardinal" Interpolation Problem (ACIP). Let $\{t_k\}_{k \in \mathbb{Z}}$ be a strictly increasing sequence of real numbers satisfying the following condition: There exist indices $h < l$ and a partition Δ (as in the cardinal spline spaces) such that $\{t_k\}_{k \geqslant h}$ agrees with $c_1 \Delta$ and $\{t_k\}_{k \leqslant l}$ agrees with $c_2 \Delta$ for arbitrary positive constants c_1 and c_2. The sequence $\delta = \{t_k\}_{k \in \mathbb{Z}}$ and the natural number r define "almost cardinal spline" spaces for $n \geqslant 2r - 1$:

$$S_{n,r,\delta} = \{f(x) \mid f_{[t_k, t_{k+1}]} \in \prod_n, f(x) \in C^{n-r}(\mathbb{R})\}.$$

Setting $\alpha = \{\alpha_k\}_{k \in \mathbb{Z}}$ as one of the two types,

(a) $\alpha_k = 0$ for all $k \in \mathbb{Z}$,

(b) $0 < \alpha_k < t_k - t_{k-1}$ for all $k \in \mathbb{Z}$ and $\alpha_k = \alpha_{k+\ell}$ for $k \geqslant h$ and $k \leqslant l$

(where δ is defined by the partition Δ), we define an Almost Cardinal Interpolation Problem on $S_{n,r,\delta}$ as: Given r bounded sequences $\{f_k^{(j)}\}_{k\in\mathbb{Z}}$, $j=0, 1, ..., r-1$, find a unique bounded function in $S_{n,r,\delta}$ satisfying $f^{(j)}(t_k+\alpha_k)=f_k^{(j)}$ for all $k\in\mathbb{Z}$ and $j=0, 1, ..., r-1$.

In [1] a wide range of problems of this type are studied, but only for n odd, and the interpolation is done at the knot sequence.

A main feature of the Cardinal Spaces $S_{n,r,\Delta}$ is their invariance under translation, which enables one to easily define the null-space associated with the interpolation problem in terms of eigensplines [4, 5, 6]—functions that have specific growth qualities as $|x|\to\infty$. Invariance under translation is not a feature of functions in spaces of almost cardinal splines, thus the characterization of the null-space associated with the ACIP is a bit more tricky. A main step in our analysis is to show that although eigensplines (in the cardinal sense) do not exist in the null-space of $S_{n,r,\delta}$, functions having similar qualities can be constructed. The existence of these functions leads us to the conclusion that the null-space of $S_{n,r,\delta}$ has the same basic structure as that of $S_{n,r,\Delta}$ in the cardinal case, and consequently the solution of an ACIP is of the same nature as that of the corresponding cardinal problem. An outline of the structure of this paper is given below.

In Section 2 we start with a characterization of the null-space of $S_{n,r,\delta}$ restricted to any interval $[t_k, t_{k+1}]$ using oscillatory matrices, and state results from the theory of oscillatory matrices needed to construct a convenient basis of the null-space.

In Section 3 we fully characterize the null-space, and give conditions for the uniqueness of a solution to the ACIP. Our main conclusion is that uniqueness is completely determined by the structure of the knot sequence δ and of the interpolating points $\{t_k+\alpha_k\}_{k\in\mathbb{Z}}$ in the "cardinal range," namely, outside the finite segment $[t_l, t_h]$.

Section 4 is devoted to the existence of a solution in those cases where uniqueness is guaranteed. This is done by showing the existence of Lagrange functions $\mathscr{L}_{k,j}(x)$ [5] and by analyzing the uniform convergence of series of these functions.

2. Preliminary Results

Following Schoenberg [5], we define the null-space of $S_{n,r,\delta}$ relative to the interpolating points $\{t_k+\alpha_k\}_{k\in\mathbb{Z}}$ as

$$S_{n,r,\delta}^{\alpha}=\{f(x)\in S_{n,r,\delta}\,|\,f^{(j)}(t_k+\alpha_k)=0,\ 0\leqslant j\leqslant r-1,\ k\in\mathbb{Z}\}.$$

The following observations about $S_{n,r,\delta}^{\alpha}$ come in handy:

(2.1) $S^{\alpha}_{n,r,\delta}$ is a finite dimensional subspace of $S_{n,r,\delta}$ of dimension μ, where

$$\mu = \begin{cases} n+1-2r, & \alpha_k = 0 \text{ for all } k \in \mathbb{Z} \\ n+1-r, & \alpha_k \neq 0 \text{ for all } k \in \mathbb{Z}. \end{cases}$$

(2.2) An element $f(x) \in S^{\alpha}_{n,r,\delta}$ is uniquely determined by its values on any interval $[t_k, t_{k+1}]$.

Due to (2.2), in order to construct a basis for $S^{\alpha}_{n,r,\delta}$ it is sufficient to construct a basis for all polynomials $p(x)$ of degree at most n satisfying for some k

$$p^{(j)}(t_k) = p^{(j)}(t_{k+1}) = 0, \qquad j = 0, 1, ..., r-1 \qquad \text{if} \quad \alpha_k = 0$$

$$p^{(j)}(t_k + \alpha_k) = 0, \qquad j = 0, 1, ..., r-1 \qquad \text{otherwise}$$

and then to extend these functions outside $[t_k, t_{k+1}]$ to elements of $S^{\alpha}_{n,r,\delta}$.

In the sequel we shall make extensive use of the following notation for $f(x)$ with l derivatives at x_0: $[f]^l_r(x_0) = (f^{r}(x_0), ..., f^{l}(x_0))^T$.

THEOREM 2.1 [4, 6]. (a) *The space of all polynomials $p(x)$ of degree at most n that satisfy the conditions: $p^{(j)}(0) = p^{(j)}(1) = 0$ for $0 \leqslant j \leqslant r-1$ is of dimension $n+1-2r$, and is spanned by polynomials $\{p_{j,r}\}_{j=1}^{n+1-2r}$ satisfying*

$$[p_{j,r}]^{n-r}_r(1) = A_r[p_{j,r}]^{n-r}_r(0) = \lambda_j [p_{j,r}]^{n-r}_r(0), \tag{2.3}$$

where $(-1)^r A_r$ is an oscillatory matrix of order $n+1-2r$.

(b) *The space of all polynomials $p(x)$ of degree at most n that satisfy the conditions $p^{(j)}(\alpha) = 0$ for $0 < \alpha < 1$ and $0 \leqslant j \leqslant r-1$ is of dimension $n+1-r$ and is spanned by polynomials $\{p^{\alpha}_{j,r}\}_{j=1}^{n+1-r}$ satisfying the equations*

$$[p^{\alpha}_{j,r}]^{n-r}_0(1) = A^{(\alpha)}_r[p^{\alpha}_{j,r}]^{n-r}_0(0) = \lambda^{(\alpha)}_j [p^{\alpha}_{j,r}]^{n-r}_0(0), \tag{2.4}$$

where $(-1)^r A^{(\alpha)}_r$ is oscillatory of the order $n+1-r$.

Remarks. (1) A rectangular matrix is called totally positive (non-negative) if all its minors of any order are positive (non-negative). A square matrix S is called oscillatory if it is totally non-negative, and there exists a positive integer k such that S^k is totally positive.

(2) It is clear from (2.3) and (2.4) that the vectors $[p_{j,r}]^{n-r}_r(0)$ are eigenvectors of A_r associated with the cigenvalues λ_j, likewise for $[p^{\alpha}_{j,r}]^{n-r}_0(0)$, the numbers $\lambda^{(\alpha)}_j$ and the matrix $A^{(\alpha)}_r$.

(3) Theorem 2.1 was stated for the interval $[0, 1]$. It is clear that this theorem holds, with obvious changes, for any interval $[t_k, t_{k+1}]$. In particular, the matrix A_r for $[t_k, t_{k-1}]$ is similar to that for $[0, 1]$.

It follows from Theorem 2.1, observation (2.2), and $S^{\alpha}_{n,r,\delta} \subset C^{n-r}$ that

COROLLARY 2.1. *There exists a basis of* $S^{\alpha}_{n,r,\Delta}$, $\{p_j(x)\}_{j=1}^{\mu}$, *with the property*

$$[p_j]_s^{n-r}(k)=\lambda_j^k\,[p_j]_s^{n-r}(0), \qquad k\in\mathbb{Z}, \quad j=1,2,\ldots,\mu$$

where

$$s=\begin{cases}0, & \alpha_k\neq 0, \quad k\in\mathbb{Z}\\ r, & \alpha_k=0, \quad k\in\mathbb{Z}\end{cases}$$

and $\lambda_1,\lambda_2,\ldots,\lambda_\mu$ *are the eigenvalues of a matrix* A *with the property*

$$[f]_s^{n-r}(k+1)=A[f]_s^{n-r}(k), \qquad k\in\mathbb{Z}, \quad f\in S^{\alpha}_{n,r,\Delta}.$$

A *can be factored into* $A_{\ell}\cdots A_1$ *where* $(-1)^r A_i$ *is an oscillatory matrix as in Theorem* 2.1 *with the property*

$$[f]_s^{n-r}(k+\xi_i)=A_i[f]_s^{n-r}(k+\xi_{i-1}), \qquad k\in\mathbb{Z}, \quad f\in S^{\alpha}_{n,r,\Delta}.$$

The following properties of oscillatory matrices due to Gantmakher and Krien [2] are of great importance to us in the forthcoming analysis.

THEOREM 2.2 [2]. *The eigenvalues of an oscillatory matrix are all real, positive, and simple.*

THEOREM 2.3 [2]. *The product of* $\mu-1$ *oscillatory matrices of order* μ *is totally positive. Furthermore the product of an oscillatory matrix and a totally positive matrix is totally positive.*

Let $\bar{x}\in\mathbb{R}^n$. We denote by $S^-(\bar{x})$ the number of sign changes in the sequence $(x_1,x_2,\ldots,x_n)$ where we ignore all zero entries. The maximum number of sign changes in $(x_1,x_2,\ldots,x_n)$ when zero entries are given either a "+" or a "−" sign is denoted by $S^+(\bar{x})$. It is clear that $S^-(\bar{x})\leqslant S^+(\bar{x})$, and a necessary condition for equality is that $x_n\neq 0$ and $x_1\neq 0$. With these notations we have:

THEOREM 2.4 [3]. *If* A *is a totally positive matrix of order* n, *then for every* $0\neq\bar{x}\in\mathbb{R}^n$ *we have* $S^+(A\bar{x})\leqslant S^-(\bar{x})$.

THEOREM 2.5 [2]. *If* A *is an oscillatory matrix of order* μ *with eigenvalues* $\lambda_1>\lambda_2>\cdots>\lambda_\mu>0$ *and associated eigenvectors* $\{x^j\}_{j=1}^{\mu}$, *then for every non-trivial vector* $\bar{x}=\sum_{j=p}^{q}\alpha_j\bar{x}^j$ *we have*

$$p-1\leqslant S^-(\bar{x})\leqslant S^+(\bar{x})\leqslant q-1.$$

In particular, the rth eigenvector has exactly $r-1$ *sign changes.*

Since we are mainly interested in the sign changes of the eigenvectors of the matrices we are dealing with, we make the following definition.

DEFINITION 2.6. (a) An oscillation matrix A is a matrix such that either A or $-A$ is oscillatory.

(b) A regulating matrix B is a matrix such that either B or $-B$ is totally positive.

This is a special case of a strictly sign-consistent (SSC) matrix as defined by Karlin [3].

Theorems 2.2–2.5 are also valid for oscillation and regulating matrices provided we order the eigenvalues of the matrix in decreasing absolute value.

3. THE NULL-SPACE $S^{\alpha}_{n,r,\delta}$

As seen in observation (2.2), in order to construct a basis for $S^{\alpha}_{n,r,\delta}$ we construct a basis for $S^{\alpha}_{n,r,\delta}$ restricted to an interval $[t_k, t_{k+1}]$. Applying Theorem 2.1 to the intervals $[t_{l-1}, t_l]$ and $[t_h, t_{h+1}]$ and extending the corresponding basis functions to elements to $S^{\alpha}_{n,r,\delta}$ we arrive at two bases of $S^{\alpha}_{n,r,\delta}$, $\{p_j(x)\}_{j=1}^{\mu}$ and $\{q_j(x)\}_{j=1}^{\mu}$, where $\mu = \dim S^{\alpha}_{n,r,\delta}$. In view of Corollary 2.1, Theorem 2.2, and the fact that the knot sequence and interpolation points are "cardinal" for $x \leqslant t_l$ and $x \geqslant t_h$, the functions $\{p_j(x)\}_{j=1}^{\mu}$ and $\{q_j(x)\}_{j=1}^{\mu}$ have the following properties:

There exist $|\lambda_1| > \cdots > |\lambda_\mu|$ and positive constants c_1 and c_2 such that

$$p_j(x) = 0(|\lambda_j|^{x/c_1}) \qquad \text{as} \quad x \to -\infty \tag{3.1}$$

$$q_j(x) = 0(|\lambda_j|^{x/c_2}) \qquad \text{as} \quad x \to \infty. \tag{3.2}$$

Note that $\lambda_1, \lambda_2, ..., \lambda_\mu$ are the same for the two basis in view of remark (3) and the structure of the space $S^{\alpha}_{n,r,\delta}$ for $x \leqslant t_l$ and $x \geqslant t_h$. In the forthcoming analysis each basis is used in one ray of $\mathbb{R}$; namely, every $f \in S^{\alpha}_{n,r,\delta}$ is represented in two forms:

$$f(x) = \sum_{j=1}^{\mu} a_j p_j(x) \qquad \text{for} \quad x < t_l \tag{3.3}$$

$$f(x) = \sum_{j=1}^{\mu} b_j q_j(x) \qquad \text{for} \quad x > t_h. \tag{3.4}$$

The next important lemma relates the representations (3.3) and (3.4) of certain elements of $S^{\alpha}_{n,r,\delta}$.

LEMMA 3.1. *Let* $0 \not\equiv f(x) \in S^{\alpha}_{n,r,\delta}$. *If in terms of the basis* $\{p_j(x)\}_{j=1}^{\mu}$, $f(x)$ *has a representation*

$$f(x) = \sum_{j=1}^{t} a_j p_j(x), \qquad t \leqslant \mu,$$

then in terms of the basis $\{q_j(x)\}_{j=1}^{\mu}$, $f(x)$ *has a representation*

$$f(x) = \sum_{j=1}^{\mu} b_j q_j(x),$$

with $b_{j_0} \neq 0$ *for some* $1 \leqslant j_0 \leqslant t$.

Proof. By the construction of the basis $\{p_j(x)\}_{j=1}^{\mu}$, $[p_j]_s^{n-r}(t_\ell)$ is an eigenvector of an oscillation matrix A associated with its jth eigenvalue in absolute value (here A and s depend on n, r, the knot sequence, and the interpolation points $\{t_k + \alpha_k\}$). Now let

$$f(x) = \sum_{j=1}^{t} a_j p_j(x), \qquad t \leqslant \mu;$$

then by Theorem 2.5

$$S^-([f]_s^{n-r}(t_\ell)) \leqslant S^+([f]_s^{n-r}(t_\ell)) \leqslant t-1, \tag{3.5}$$

while by Theorem 2.1 we have $[f]_s^{n-r}(t_{k+1}) = A_k [f]_s^{n-r}(t_k)$, where A_k is the oscillation matrix of the theorem. Thus for every $m \geqslant 1$ we have

$$[f]_s^{n-r}(t_{\ell+m}) = \left(\prod_{i=k}^{k+m-1} A_i\right) [f]_s^{n-r}(t_\ell)$$

and for m sufficiently large, by Theorem 2.3, $(\prod_{i=\ell}^{\ell+m-1} A_i)$ is regulating; hence, by Theorem 2.4,

$$S^+[f]_s^{n-r}(t_{\ell+m}) \leqslant S^-([f]_s^{n-r}(t_\ell)). \tag{3.6}$$

Combining (3.5) with (3.6) we conclude that for k large enough

$$S^-([f]_s^{n-r}(t_k)) \leqslant S^+([f]_s^{n-r}(t_k)) \leqslant S^-([f]_s^{n-r}(t_\ell)) \leqslant t-1.$$

In particular, there exists a $k^* \geqslant h$ such that

$$S^-([f]_s^{n-r}(t_k)) \leqslant S^+([f]_s^{n-r}(t_k)) \leqslant t-1, \qquad k \geqslant k^* \geqslant n. \tag{3.7}$$

However, $f(x)$ can be represented as $f(x) = \sum_{j=1}^{\mu} b_j q_j(x)$, where, by Corollary 2.1 and the construction of $\{q_j(x)\}_{j=1}^{\mu}$, $[q_j]_s^{n-r}(t_{n+k\ell})$ for $k \geqslant 0$ are eigenvectors of an oscillation matrix corresponding to the jth eigenvalue. Thus the assumption $b_j = 0$ for $j \leqslant t$ leads in view of Theorem 2.5 to the inequalities $t \leqslant S^-([f]_s^{n-r}(t_m)) \leqslant S^+([f]_s^{n-r}(t_m))$ for $m = h + k\tilde{c}$, $k \geqslant 0$, which clearly contradicts (3.7).

From this point our analysis depends on the values of $\lambda_1, \lambda_2, ..., \lambda_\mu$ in (3.1) and (3.2). We consider the two cases:

(I) $|\lambda_j| \neq 1$ for all $1 \leqslant j \leqslant \mu$. Namely, there exists $0 \leqslant m_0 \leqslant \mu$ such that

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_{m_0}| > 1 > |\lambda_{m_0+1}| > \cdots > |\lambda_\mu|. \tag{3.8}$$

(II) There exists a unique j_0, $1 \leqslant j_0 \leqslant \mu$, such that $|\lambda_{j_0}| = 1$. In this case we have

$$|\lambda_1| > \cdots > |\lambda_{j_0}| = 1 > |\lambda_{j_0+1}| > \cdots > |\lambda_\mu| > 0. \tag{3.9}$$

We now proceed to show that in case (I) the only bounded function in $S^\alpha_{n,r,\delta}$ is the zero function, while in case (II) there exists (up to a multiplicative constant) a unique bounded function in $S^\alpha_{n,r,\delta}$.

THEOREM 3.2. *If case* (3.8) *holds then* $\dim(S^\alpha_{n,r,\delta} \cap L^\infty(\mathbb{R})) = 0$.

Proof. Assume that $0 \not\equiv f(x) \in S^\alpha_{n,r,\delta}$ is bounded on $\mathbb{R}$. Then it follows from (3.1), (3.2), and (3.8) that $f(x)$ has the two representations

$$f(x) = \sum_{j=1}^{m_0} a_j p_j(x) = \sum_{j=m_0+1}^{\mu} b_j q_j(x) \tag{3.10}$$

in contradiction to Lemma 3.1.

THEOREM 3.3. *If case* (3.9) *holds then* $\dim(S^\alpha_{n,r,\delta} \cap L^\infty(\mathbb{R})) = 1$.

Proof. Since $\dim S^\alpha_{n,r,r} = \mu$, there exists a non-zero function f common to the two subspaces of $S^\alpha_{n,r,\delta}$, $\operatorname{span}\{p_1(x), ..., p_{j_0}(x)\}$ and $\operatorname{span}\{q_{j_0}(x), ..., q_\mu(x)\}$. In view of (3.1), (3.2), and (3.9), $f \in S^\alpha_{n,r,\delta} \cap L^\infty(\mathbb{R})$ and any such f has the two representations

$$f(x) = \sum_{j=1}^{j_0} \alpha_j p_j(x) = \sum_{j=j_0}^{\mu} b_j q_j(x). \tag{3.11}$$

These representations are consistent with Lemma 3.1 if and only if

$$a_{j_0} \neq 0, \qquad b_{j_0} \neq 0. \tag{3.12}$$

Hence if $f, g \in S^\alpha_{n,r,r} \cap L^\infty(\mathbb{R})$ and are linearly independent then it is possible to choose α, β $(\alpha^2 + \beta^2 > 0)$ such that in the representation of

$\alpha f+\beta g$ in terms of the basis $\{p_j\}$ the coefficient of p_{j_0} vanishes in contradiction to (3.12). This completes the proof of the theorem.

COROLLARY 3.4. *If $f \in S^{\alpha}_{n,r,\delta} \cap L^{\infty}(\mathbb{R})$ then there exist $a \neq 0$, $b \neq 0$ such that*

$$|f(x) - ap_{j_0}(x)| = 0(|\lambda_{j_0-1}|^x) \qquad as \quad x \to -\infty \tag{3.13}$$

$$|f(x) - bq_{j_0}(x)| = 0(|\lambda_{j_0+1}|^x) \qquad as \quad x \to \infty. \tag{3.14}$$

Corollary 3.4 is the generalization to the almost cardinal case of the periodicity of $f \in S^{\alpha}_{n,r,\delta} \cap L^{\infty}(\mathbb{R})$ in the cardinal case [5].

4. EXISTENCE OF A SOLUTION TO THE ACIP

Since the unicity of the solution to our interpolation problem is completely determined by the dimension of the null-space $S^{\alpha}_{n,r,\delta}$, we are motivated by Theorems 3.2 and 3.3 to seek a solution only in case (I) when uniqueness is guaranteed. Sufficient conditions where case (I) holds can be found in [4].

PROPOSITION 4.1. *In case* (I), *with m_0 as in* (3.8), *the functions $p_1(x), ..., p_{m_0}(x), q_{m_0+1}(x), ..., q_{\mu}(x)$ constitute a basis of $S^{\alpha}_{n,r,\delta}$.*

Proof. The functions in each of the two sets $\{p_j(x)\}_{j=1}^{m_0}$ and $\{q_j(x)\}_{j=m_0+1}^{\mu}$ are clearly linearly independent and since their total number is μ we have only to show that the intersection of the spaces they span is the zero function. This follows directly from Theorem 3.2 since every function in the intersection is bounded on $\mathbb{R}$.

THEOREM 4.2. *In case* (I) *there exist functions $\mathscr{L}_{k,i}(x) \in S_{n,r,\delta}$ satisfying*

$$\mathscr{L}^{(s)}_{k,i}(t_j+\alpha_j) = \delta_{k,j} \cdot \delta_{s,i} \qquad for \quad j, k \in \mathbb{Z} \ and \ 0 \leqslant i, s \leqslant r. \tag{4.1}$$

Moreover there exist c_k and γ positive constants such that

$$|\mathscr{L}_{k,i}(x)| \leqslant c_k e^{-\gamma|x-t_k|}. \tag{4.2}$$

Proof. The existence of functions in $S_{n,r,\delta}$ that satisfy (4.1) but are not necessarily bounded is obvious. We choose a function $g_{k,i}(x)=g(x)$ in $S^{\alpha}_{n,r,\delta}$ that satisfies (4.1). The function $g(x)$ when restricted to $(-\infty, t_{k-1})$ can be looked upon as an element of the space $S^{\alpha}_{n,r,\delta}$ (restricted to $(-\infty, t_{k-1})$). Thus $g(x)$ can be represented on this interval as

$$g(x) = \sum_{j=1}^{m_0} a_j p_j(x) + \sum_{j=m_0+1}^{\mu} a_j q_j(x), \qquad x \leqslant t_{k-1}. \tag{4.3}$$

In the same fashion, restricting $g(x)$ and $S^{\alpha}_{n,r,\delta}$ to (t_{k+1}, ∞), $g(x)$ can be represented as

$$g(x) = \sum_{j=1}^{m_0} b_j p_j(x) + \sum_{j=m_0+1}^{\mu} b_j q_j(x), \qquad x \geqslant t_{k+1}. \tag{4.4}$$

The function $\mathscr{L}_{k,i}(x)$, defined as

$$\mathscr{L}_{k,i}(x) = g(x) - \left(\sum_{j=m_0+1}^{\mu} a_j q_j(x) + \sum_{j=1}^{m_0} b_j p_j(x) \right),$$

satisfies (4.1) and in view of (4.3) and (4.4) has the representations

$$\begin{aligned} \mathscr{L}_{k,i}(x) &= \sum_{j=1}^{m_0} (a_j - b_j)\, p_j(x), && x \leqslant t_{k-1}, \\ &= \sum_{j=m_0+1}^{\mu} (b_j - a_j)\, q_j(x), && x \geqslant t_{k+1}. \end{aligned}$$

Thus by (3.8) and the asymptotic behavior of $\{p_j(x)\}$ and $\{q_j(x)\}$ as stated in (3.1) and (3.2), we conclude that $\mathscr{L}_{k,i}(x)$ has properties (4.1) and (4.2) as desired.

The functions $\mathscr{L}_{k,i}(x)$ are called the Lagrange functions of $S^{\alpha}_{n,r,\delta}$. Property (4.2) of the functions $\mathscr{L}_{k,i}(x)$ is not sufficient to prove that series of the type $\sum_{k \in \mathbb{Z}} \sum_{j=0}^{r-1} f_k^{(j)} \mathscr{L}_{k,j}(x)$ converge to the interpolating function except for the case of cardinal interpolation, where the $\mathscr{L}_{k,i}(x)$ are translates of $\mathscr{L}_{0,i}(x)$. We have to strengthen (4.2) and show that $c_k \leqslant c$ for all $k \in \mathbb{Z}$.

LEMMA 4.3. *Let $k < l$. The functions $\mathscr{L}_{k,i}(x)$, whose existence was proven in Theorem 4.2, have the representation*

$$\mathscr{L}_{k,i}(x) = \bar{g}_{k,i}(x) - \sum_{j=1}^{m_0} a_{k,j,i}\, p_j(x), \tag{4.5}$$

where

$$|\bar{g}_{k,i}(x)| \leqslant c e^{\gamma(x - t_k)} \qquad \textit{for} \quad x \leqslant t_l, \tag{4.6}$$

and

$$|a_{k,j,i}| \leqslant c\beta^k \qquad \textit{for some} \quad \beta > 1. \tag{4.7}$$

Furthermore for $x \geqslant t_{k+1}$, $\mathscr{L}_{k,i}(x)$ has the representation

$$\mathscr{L}_{k,i}(x) = \sum_{j=m_0+1}^{\mu} b_{k,j,i}\, q_j(x). \tag{4.8}$$

where

$$|b_{k,j,i}| \leqslant c\beta^k \qquad \textit{for some} \quad \beta > 1. \tag{4.9}$$

Proof. Let $S^*_{n,r,\cdot}$ be the cardinal spline space whose elements are in $S^\alpha_{n,r,\delta}$ for all $x \leqslant t_l$, and define $\bar{g}_{k,i}(x)$ to be the functions in $S^\alpha_{n,r,\delta}$ that satisfy (4.1) and agree with the Lagrange functions of $S^*_{n,r,\cdot}$ for all $x \leqslant t_l$. Due to the fact that the Lagrange functions of $S^*_{n,r,\cdot}$ are translates of the functions $\{\mathscr{L}_{0,i}(x)\}_{i=0}^{r}$ and these decay exponentially by Theorem 4.2, we have (4.6).

As seen in the proof of Theorem 4.2 for $x \geqslant t_{k+1}$, $\bar{g}_{k,i}(x)$ can be represented as

$$\bar{g}_{k,i}(x) = \sum_{j=1}^{m_0} a_{k,j,i}\, p_j(x) + \sum_{j=m_0+1}^{\mu} b_{k,j,i}\, q_j(x). \tag{4.10}$$

Examining $\bar{g}_{k,i}(x)$ on $[t_{l-1}, t_l]$ and taking $k \to -\infty$, we see from (4.6) that $\operatorname{Max}_{x\in[t_{l-1},t_l]} |\bar{g}_{k,i}(x)| \leqslant c\beta^k$ for some $\beta > 1$ and since $\{p_1, ..., p_{m_0}, q_{m_0+1}, ..., q_\mu\}$ is a basis for $S^\alpha_{n,r,\delta}$, we conclude that

$$|a_{k,j,i}| \leqslant C\beta^k \qquad \text{and} \qquad |b_{k,j,i}| \leqslant C\beta^k \tag{4.11}$$

for the appropriate j's and i's.

As seen in Theorem 4.2 in order to obtain $\mathscr{L}_{k,i}(x)$ from $\bar{g}_{k,i}(x)$ we subtract a suitable element of $S^\alpha_{n,r,\delta}$. In our case $\bar{g}_{k,i}(x)$ already has the desired exponential decay as $x \to -\infty$, thus defining $\mathscr{L}_{k,i}$ as in (4.5) we conclude from (4.11) the estimate (4.7). Moreover (4.8) and (4.9) follow from (4.5), (4.10), and (4.11). ∎

We are now in a position to prove that:

THEOREM 4.4. *There exist positive constants c and γ (independent of k) such that the Lagrange functions $\mathscr{L}_{k,i}(x)$ satisfy*

$$|\mathscr{L}_{k,i}(x)| \leqslant c e^{-\gamma|x-t_k|}. \tag{4.12}$$

Proof. We prove this only for $k < l$. For $k > h$ the proof is similar, while the functions $\mathscr{L}_{k,i}(x)$ for $l \leqslant k \leqslant h$ are finite in number, satisfy (4.2), and thus do not affect this result. The proof of (4.12) for $k < l$ is done by estimating $\mathscr{L}_{k,i}(x)$ on different intervals.

(I) For $x \geqslant t_h$, $\mathscr{L}_{k,i}(x)$ satisfy (4.12).

We use (4.8), and show that every term in the sum satisfies (4.12). For $m_0 + 1 \leqslant j \leqslant \mu$, $\{q_j(x)\}$ decay exponentially as $x \to \infty$, thus for $x \geqslant t_h$, $|q_j(x)| \leqslant ce^{-\gamma_2 x}$. Moreover $|b_{k,j,i}| \leqslant c\beta^k$ for some $\beta > 1$. Thus $|b_{k,j,i}q_j(x)| \leqslant \beta^k c_2 e^{-\gamma_2 x} \leqslant c_1 e^{-\gamma_1(x-t_k)}$. But $x \geqslant t_k$ and thus $(x - t_k) = |x - t_k|$ and (4.1) follows.

(II) For $x < t_l$, $\mathcal{L}_{k,i}(x)$ satisfies (4.12).

For this range we use the representation (4.5) and the estimate (4.6). Since $\bar{g}_{k,i}(x)$ satisfies (4.6) in this interval it remains to estimate every term in the sum $\sum_{j=1}^{m_0} a_{k,j,i} p_j(x)$. For $1 \leq j \leq m_0$, $\{p_j(x)\}$ decay exponentially as $x \to -\infty$, and by (4.7) the $a_{k,j,i}$ decay as $k \to -\infty$. Thus $|a_{k,j,i} p_j(x)| \leq Ce^{-\gamma|x-t_k|}$ for similar constants which clearly implies (4.12).

(III) For $x \in (t_l, t_h)$, $\mathcal{L}_{k,i}(x)$ satisfies (4.12).

The segment $t_l \leq x \leq t_h$ is finite and $q_{m_0+1}, ..., q_n$ are bounded there. Thus by (4.8) and (4.11)

$$|\mathcal{L}_{k,i}(x)| \leq A\beta^k \tag{4.13}$$

for an appropriate constant A. This leads to (4.12).

THEOREM 4.5. *Suppose* (3.8) *holds. Then for every* r *bounded sequences* $\{f_k^{(j)}\}_{k \in \mathbb{Z}}$, $0 \leq j \leq r-1$, *there exists a unique bounded function* $f(x) \in S_{n,r,\alpha}$ *such that* $f^{(j)}(t_k + \alpha_k) = f_k^{(j)}$ *for all* $k \in \mathbb{Z}$ *and* $0 < j \leq r-1$.

Proof. Let $\mathcal{L}_{k,j}(x)$ be the Lagrange functions of Theorem 4.4. Then the formal series

$$\sum_{k \in \mathbb{Z}} \sum_{j=0}^{r-1} f_k^{(j)} \mathcal{L}_{k,j}(x) \tag{4.14}$$

clearly interpolates the data $\{f_k^{(j)}\}$, $k \in \mathbb{Z}$, $0 \leq j \leq r-1$.

It is elementary to see that (4.14) converges uniformly on compact subsets of $\mathbb{R}$. Since (4.14) is in every segment $[t_k, t_{k+1}]$ a series of polynomials of degree at most n, it converges to an element of $S_{n,r,\alpha}^{\infty}$. Finally due to the boundedness of $\{f_k^{(j)}\}$ and to property (4.12) of $\mathcal{L}_{k,j}(x)$ we conclude that (4.14) represents a bounded element of $S_{n,r,\alpha}^{\infty}$.

REFERENCES

1. C. DE BOOR, Odd-degree spline interpolation at a bifinite knot sequence, *in* "Approximation Theory, Bonn 1976," Lecture Notes Math., Vol. 556, pp. 30–53, Springer-Verlag, Heidelberg, 1976.
2. F. R. GANTMAKHER AND M. G. KREIN, "Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems," 2nd ed., State Publishing House, Moscow, 1950.
3. S. KARLIN, "Total Positivity," Stanford Univ. Press, Stanford, CA, 1968.
4. C. A. MICCHELLI, Oscillation matrices and cardinal spline interpolation, *in* "Spline Functions and Approximation Theory," pp. 163–201, Academic Press, New York, 1978.
5. I. J. SCHOENBERG, "Cardinal Spline Interpolation," CBMS 12, SIAM, Philadelphia.
6. I. J. SCHOENBERG, On Micchelli's theory of cardinal L-splines, *in* "Spline Functions and Approximation Theory," pp. 202–276, Academic Press, New York, 1978.